Introduction to Particle Accelerator Physics

Tutorial 2 - Solutions

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1. Thin Lens Approximation

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty}(-1)^n \tfrac{x^{2n}}{(2n)!} &&= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - + \dots \\ \sin x &= \sum_{n=0}^{\infty}(-1)^n \tfrac{x^{2n+1}}{(2n+1)!} &&= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - + \dots\end{aligned}$$

$$M_Q = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix} = \begin{pmatrix} 1+\mathcal{O}(L^2) & \frac{\sqrt{k}L}{\sqrt{k}} + \mathcal{O}(L^3) \\ -kL + \mathcal{O}(L^3) & 1+\mathcal{O}(L^2) \end{pmatrix}$$

In the thin lens approximation we assume

$$kL = \frac{1}{f} = \lim_{L\to 0}\int_0^{\infty} k(s)\,ds = const$$

$$\Longrightarrow \lim_{L\to 0} M_Q = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix}$$

2. Drift Sections and Quadrupole Doublets

a)

$$M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$x = x_0 + Lx_0' \quad \Longrightarrow \quad x_0' = \frac{x - x_0}{L}$$

b)

$$M = M_{QD}M_DM_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{L}{f^2} & 1+\frac{L}{f} \end{pmatrix}$$

$$\Longrightarrow x = \left(1 - \frac{L}{f}\right)x_0 + Lx_0'$$

If we tune the quadrupole focusing strength so that $\frac{L}{f} = 1$ we get $x = Lx_0'$ which is independent of x_0.

c) From the lecture recall the definition of the quadrupole strength: $k = \frac{e}{p}g$ where g is the focusing gradient of the quadrupole. In 2b) we required $L = f$ and from the lecture we recall $\frac{1}{f} = k \cdot L$. Putting everything together we get

$$g = \frac{p}{e} \cdot k = \frac{p}{e} \cdot \frac{1}{Lf} = \frac{p}{e} \cdot \frac{1}{L^2} = \frac{0.65 \text{ GeV/c}}{0.29979} \cdot \frac{1}{(0.931 \text{ mm})^2} = 2.5 \text{ T/m}$$

3. Hill's Equation

a) Hill's equation is

$$x'' + k(s) \cdot x = 0$$

Here we assume $k(s) = k = \text{const}$

$$x'' + kx = 0 \quad \Longrightarrow \quad x'' = -kx$$

This is the ordinary differential equation (ODE) of a harmonic oscillator (HO).

b) For simplicity we introduce the notation $x(0) = x_0$ and $x'(0) = x'_0$. The general (real) solution of the ODE $x'' = -kx$ is given by

$$x(s) = x_0 \cdot \cos\sqrt{k}s \; + \; \frac{x'_0}{\sqrt{k}} \cdot \sin\sqrt{k}s$$

This is a harmonic oscillation for $\mathbf{k > 0}$.
If $\mathbf{k < 0}$ we make use of the identity $\sqrt{k} = i\sqrt{|k|}$

$$x(s) = x_0 \cdot \cosh\sqrt{|k|}s \; + \; \frac{x'_0}{\sqrt{|k|}} \cdot \sinh\sqrt{|k|}s$$

This is exponential damping or excitation.
If $\mathbf{k = 0}$ the ODE becomes $x'' = 0$ with the general solution

$$x(s) = x_0 + x'_0 s$$

This is linear propagation.

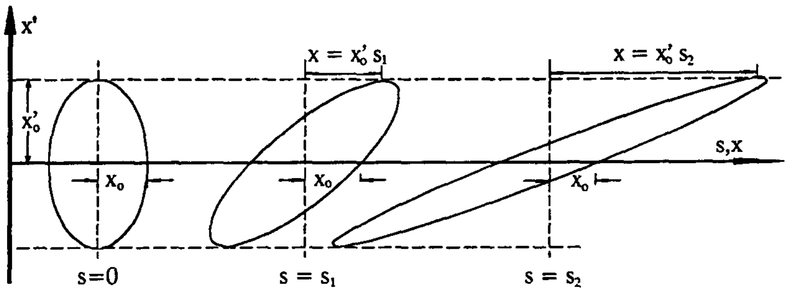
c) Within each interval i where $k(s) = k_i$ is constant the solution of Hill's equation is the solution of a harmonic oscillator with $k = k_i$. If the solving functions $x_i(s)$ are combined to a smooth function $x(s)$ one has an approximation for the solution of the original Hill's equation.

4. Evolution of a Phase Space Ellipse

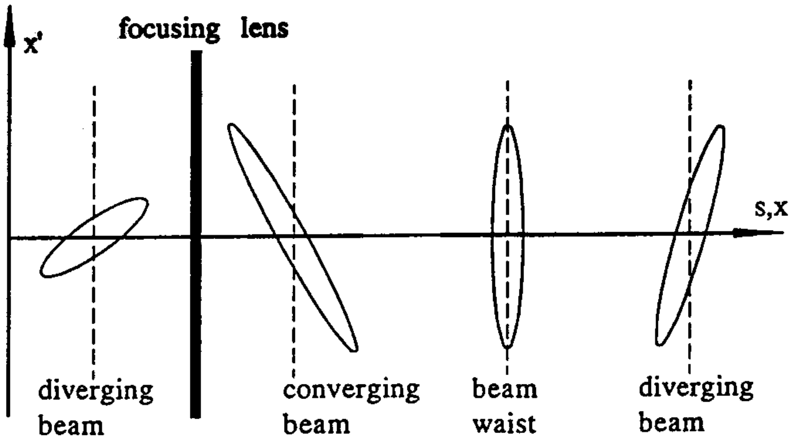
a) From the transport matrix of a drift section we gather

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M_D \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} x_0 + x'_0 \cdot s \\ x'_0 \end{pmatrix}$$

So the divergence $x' = x'_0$ remains constant while the beam size $x = x_0 + x'_0 \cdot s$ increases linearly with s.



b) The initially diverging beam becomes convergent after passing the focussing lens. It converges until it reaches the beam waist where the beam size reaches a minimum (and the divergence reaches a maximum). From here on the beam becomes divergent again.



5. Transformation of the Beta Function

a) The beta matrix transforms like

$$B_s = M B_0 M^T$$

Here we assume

$$B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$$

and we know

$$M = M_D = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Together this leads to

$$B_s = \begin{pmatrix} \beta_s & -\alpha_s \\ -\alpha_s & \gamma_s \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & \frac{s}{\beta_0} \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix}$$

So we end up with

$$\begin{aligned} \beta(s) &= \beta_0 + \frac{s^2}{\beta_0} \\ \alpha(s) &= -\frac{s}{\beta_0} \\ \gamma(s) &= \frac{1}{\beta_0} \end{aligned}$$

b) The Twiss parameters transform like

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

where C denotes the cosine-like function and S denotes the sine-like function in the transformation matrix. In the thin lens approximation we can write

$$M = M_D = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

This then gives us

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} \\ -\frac{s}{\beta_0} \\ \frac{1}{\beta_0} \end{pmatrix}$$

which is identical to the results we derived in 4a).

c) The betatron function $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$ in the vicinity of a symmetry point in a drift section grows quadratically with the distance s from the symmetry point. This growth is larger for smaller values of β_0. This is a direct consequence of the Liouville's theorem: the area in phase space can not be reduced by a focussing element, thus a reduction of transverse beam size leads to a simultaneous increase of beam divergence. This is illustrated below ($\beta^\star$ denotes β_0).