Introduction to Particle Accelerator Physics

Solutions to Exercise 5

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1. Apertures, Dispersion, and Acceptance

a) Recall from the lecture the definition of the vertical acceptance (without dispersion)

$$A_y = \min\left(\frac{a_y(s)^2}{\beta_y(s)}\right)$$

where $a_y(s)$ is the vertical acceptance. In this case the vertical acceptance is constant around the ring so the minimum will be reached when the vertical beta function reaches a maximum:

$$A_y = \min\left(\frac{a_y(s)^2}{\beta_y(s)}\right) = \frac{(h/2)^2}{\beta_{y,\max}} = \frac{(15\,\mathrm{mm})^2}{25\,\mathrm{m/rad}} = 9\,\mathrm{mm\,mrad}$$

b) Since we know there is a vertical focus at the center of the straight (symmetry point), we can refer to the result of exercise 4 on exercise sheet 2 for the beta function around a symmetry point:

$$\beta_y(s) = \beta_{y,0} + \frac{s^2}{\beta_{y,0}}$$

From this we gather that the beam will be the widest at the edges of the undulator and therefore this is where we exepct the acceptance limitation:

$$\beta_{y,\mathrm{edge}} = \beta_{y,0} + \frac{(L/2)^2}{\beta_{y,0}} = 1\,\mathrm{m/rad} + \frac{(2\,\mathrm{m})^2}{1\,\mathrm{m/rad}} = 5\,\mathrm{m/rad}$$

The vertical aperture of the undulator is one half of the gap height which leads to

$$A_y = \min\left(\frac{a_y(s)^2}{\beta_y(s)}\right) = \frac{(g_u/2)^2}{\beta_{y,\mathrm{edge}}} = \frac{(3\,\mathrm{mm})^2}{5\,\mathrm{m/rad}} = 1.8\,\mathrm{mm\,mrad}$$

So the installation of the narrow-gap undulator has reduced the vertical acceptance by a factor 5!

c) Remember the definition of the betatron phase advance:

$$\phi(0,l) = \int_0^l \frac{ds}{\beta(s)}$$

This allows us to calculate the total phase advance for the undulator:

$$\begin{aligned}
\phi_{x,y} &= \int_{-L/2}^{+L/2} \frac{1}{\beta_{x,y}(s)} ds \\
&= \int_{-L/2}^{+L/2} \frac{1}{\beta_{x,y,0} + \frac{s^2}{\beta_{x,y,0}}} ds \\
&= \beta_{x,y,0} \int_{-L/2}^{+L/2} \frac{1}{\beta_{x,y,0}^2 + s^2} ds \\
&= \left[\arctan \frac{s}{\beta_{x,y,0}}\right]_{s=-L/2}^{s=+L/2} \\
&= 2\arctan \frac{L/2}{\beta_{x,y,0}} \qquad (1)
\end{aligned}$$

So we can calculate the total phase advance in x and y:

$$\phi_x = 2\arctan\frac{L/2}{\beta_{x,0}} = 2\arctan\frac{2\,\mathrm{m}}{9\,\mathrm{m/rad}} \approx 25°$$

$$\phi_y = 2\arctan\frac{L/2}{\beta_{y,0}} = 2\arctan\frac{2\,\mathrm{m}}{1\,\mathrm{m/rad}} \approx 127°$$

d)

$$A_y = \min\left(\frac{a_y(s)^2}{\beta_y(s)}\right) = \min\left(\frac{(g_u/2)^2}{\beta_y(s)}\right) = \frac{(g_u/2)^2}{\hat{\beta}_y}$$

where $\hat{\beta}_y$ is the maximum of the beta function at the edge of the undulator. We can calculate the optimum $\hat{\beta}_{y,0}$ which minimizes the beta function at the undulator edge:

$$\beta_{y,edge} = \beta_{y,0} + \frac{(L/2)^2}{\beta_{y,0}}$$

$$\frac{d\beta_{y,edge}}{d\beta_{y,0}} = 1 - \frac{(L/2)^2}{\beta_{y,0}^2}$$

$$0 \stackrel{!}{=} \frac{d\beta_{y,edge}}{d\beta_{y,0}} = 1 - \frac{(L/2)^2}{\hat{\beta}_{y,0}^2}$$

Which we then further simplify:

$$\hat{\beta}_{y,0}^2 = (L/2)^2$$

$$\implies \hat{\beta}_{y,0} = \pm L/2$$

For this optimum $\hat{\beta}_{y,0}$ the beta function at the edges of the undulator is minimized:

$$\hat{\beta}_y = \hat{\beta}_{y,0} + \frac{(L/2)^2}{\hat{\beta}_{y,0}} = 2\,\mathrm{m/rad} + \frac{(2\,\mathrm{m})^2}{2\,\mathrm{m/rad}} = 4\,\mathrm{m/rad}$$

And thus the acceptance of the undulator is maximized:

$$\hat{A}_y = \frac{(g_u/2)^2}{\hat{\beta}_y} = \frac{(3\,\mathrm{mm})^2}{4\,\mathrm{m/rad}} = 2.25\,\mathrm{mm\,mrad}$$

So by properly choosing the parameter $\beta_{y,0}$, we gain 25% in acceptance!

e) Since we require the bending magnets to do the vertical focussing, we can expect the maximum of the vertical beta function $\beta_{y,\max}$ to be inside the bending magnet. We can then take the optimum acceptance $\hat{A}_y$ from above and ask what minimum aperture can be chosen in the bending magnet without limiting the vertical acceptance:

$$\hat{a}_y = \sqrt{\hat{A}_y \cdot \beta_{y,\max}} = \sqrt{2.25\,\mathrm{mm}\cdot\mathrm{mrad}\,\cdot\,25\,\mathrm{m}} = 7.5\,\mathrm{mm}$$

So, without limiting the acceptance in the bending magnet the required gap height would be only (2×4 mm required for the vacuum chamber thickness):

$$\hat{g}_m = 2\times 4\,\mathrm{mm} + 2\hat{a}_y = 23\,\mathrm{mm}$$

Compared to the originally planned 38 mm of required magnetic gap height, this is a 40% reduction!

f) Assuming constant current density, we see that for the required current:

$$I = j \cdot A \propto j \cdot g_m \propto g_m$$

From Ohm's law we get the power $P = RI^2$ and if we consider that the ohmic resistance scales inversely with the magnetic gap $R \propto g_m^{-1}$ we can derive:

$$P = RI^2 \propto g_m^{-1}\, g_m^2 \propto g_m$$

i.e. the power scales linearly with the magnetic gap. Therefore, a 40% reduction of magnetic gap height corresponds to a 40% decrease in power consumption!

2. Solenoid Focussing

a) The solenoid transfer matrix as defined in the lecture:

$$M_{sol} = \begin{pmatrix} \frac{1}{2}(1+\cos\varphi) & \rho\sin\varphi & \frac{1}{2}\sin\varphi & \rho(1-\cos\varphi) \\ -\frac{1}{4\rho}\sin\varphi & \frac{1}{2}(1+\cos\varphi) & -\frac{1}{4\rho}(1-\cos\varphi) & \frac{1}{2}\sin\varphi \\ -\frac{1}{2}\sin\varphi & -\rho(1-\cos\varphi) & \frac{1}{2}(1+\cos\varphi) & \rho\sin\varphi \\ \frac{1}{4\rho}(1-\cos\varphi) & -\frac{1}{2}\sin\varphi & -\frac{1}{4\rho}\sin\varphi & \frac{1}{2}(1+\cos\varphi) \end{pmatrix}$$

We can now evaluate the matrix for different values of $\varphi = \frac{L}{\rho} = \frac{e}{p}BL$:

$$(M_{sol})_{\varphi=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the identity matrix which is what we would expect for a solenoid with $\varphi = 0$ (either length $L = 0$ or field strength $B = 0$). For $\varphi = \pi/2$ we get:

$$(M_{sol})_{\varphi=\frac{\pi}{2}} = \begin{pmatrix} 1/2 & \rho & 1/2 & \rho \\ -1/4\rho & 1/2 & -1/4\rho & 1/2 \\ -1/2 & -\rho & 1/2 & \rho \\ 1/4\rho & -1/2 & -1/4\rho & 1/2 \end{pmatrix}$$

Which looks similar to a combination of a drift space and a focussing lens (observe the (x, x') and (y, y') sub-matrices!). And finally for $\varphi = \pi$:

$$(M_{sol})_{\varphi=\pi} = \begin{pmatrix} 0 & 0 & 0 & 2\rho \\ 0 & 0 & -1/2\rho & 0 \\ 0 & -2\rho & 0 & 0 \\ 1/2\rho & 0 & 0 & 0 \end{pmatrix}$$

For this solenoid tuning the divergences reduce to: $x' = -y_0/2\rho$ and $y' = x_0/2\rho$, i.e. the divergences at the end of this solenoid structure are independent of the initial divergences and only depend on the initial beam size!

b) In order to fully focus the beam we will require $\varphi \stackrel{!}{=} \pi$ where

$$\varphi = \frac{L}{\rho} = \frac{e}{p}BL$$

Therefor the required field strength on axis is:

$$B = \frac{\varphi}{L} \cdot \frac{p}{e} = \frac{\pi}{L} \cdot \frac{m_o c\beta\gamma}{e} \approx 176\,\mathrm{mT}$$