Introduction to Particle Accelerator Physics

Solutions to Exercise 4

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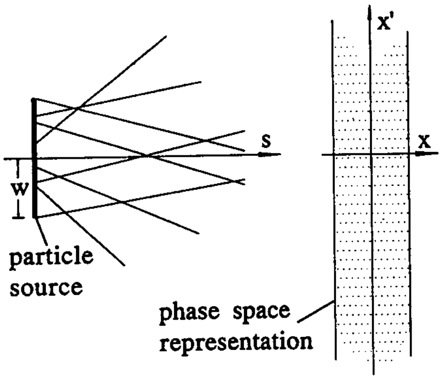
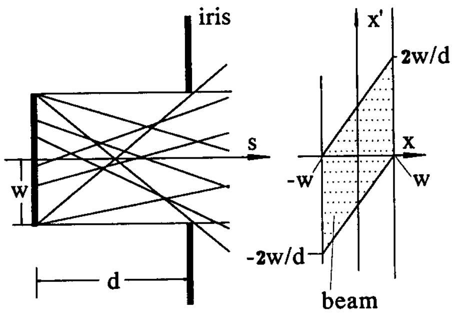
1. Phase Space Representations of Particle Sources

a) Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\phi \in [-\pi/2, +\pi/2]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



b) Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.



2. Quadrupole Errors and Tune Shifts

From the lecture recall the one-turn matrix at an arbitrary location:

$$M = \begin{pmatrix} \cos 2\pi Q + \alpha \sin 2\pi Q & \beta \sin 2\pi Q \\ -\gamma \sin 2\pi Q & \cos 2\pi Q - \alpha \sin 2\pi Q \end{pmatrix}$$

Assume now that at this location a very small gradient error is applied to the otherwise undisturbed optics (denoted by subscript $_0$):

$$\hat{M} = \begin{pmatrix} 1 & 0 \\ -\Delta(kl) & 1 \end{pmatrix} \cdot M_0$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & 0 \\ -\Delta(kl) & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\pi Q_0 + \alpha_0 \sin 2\pi Q_0 & \beta_0 \sin 2\pi Q_0 \\ -\gamma_0 \sin 2\pi Q_0 & \cos 2\pi Q_0 - \alpha_0 \sin 2\pi Q_0 \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\pi Q_0 + \alpha_0 \sin 2\pi Q_0 & \beta_0 \sin 2\pi Q_0 \\ -\gamma_0 \sin 2\pi Q_0 - \Delta(kl) \cos 2\pi Q_0 - \Delta(kl)\alpha_0 \sin 2\pi Q_0 & \cos 2\pi Q_0 - \alpha_0 \sin 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0 \end{pmatrix}
\end{aligned}
$$

In order to investigate the new tune $Q = Q_0 + \Delta Q$ we will compare the traces of the matrices:

$$
\begin{aligned}
Tr(M) &= Tr(\hat{M}) \\
2\cos 2\pi Q &= 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0
\end{aligned}
$$

We keep in mind that $Q = Q_0 + \Delta Q$ and make use of a trigonometric identity to rewrite the left hand side:

$$
2\cos 2\pi Q_0 \cos 2\pi\Delta Q - 2\sin 2\pi Q_0 \sin 2\pi\Delta Q = 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0
$$

We recall the assumption that the tune shift will be small $\Delta Q \ll 1$ which allows us to apply the two Taylor approximations $\cos 2\pi\Delta Q \approx 1$ and $\sin 2\pi\Delta Q \approx 2\pi\Delta Q$:

$$
2\cos 2\pi Q_0 - 2\pi\Delta Q \, 2\sin 2\pi Q_0 = 2\cos 2\pi Q_0 - \Delta(kl)\beta_0 \sin 2\pi Q_0
$$

Which then gives us:

$$
4\pi\Delta Q \sin 2\pi Q_0 = \Delta(kl)\beta_0 \sin 2\pi Q_0
$$

Resulting in the tune shift:

$$
\Delta Q = \frac{1}{4\pi}\beta_0 \Delta(kl)
$$

3. Momentum Compaction and Transition Energy

From the lecture recall the definition of the momentum compaction factor:

$$
\frac{\Delta L}{L} = \alpha_c \cdot \frac{\Delta p}{p}
$$

In order to look at changes in period length ΔT we have to keep in mind how T and L are related and make use of the logarithmic derivative:

$$
\begin{aligned}
T &= \frac{L}{c\beta} \\
\log T &= \log L - \log c\beta \\
\Longrightarrow \frac{dT}{T} &= \frac{dL}{L} - \frac{d\beta}{\beta}
\end{aligned}
$$

In order to plug this together with the definition of the momentum compaction factor, we need to investigate $\frac{d\beta}{\beta}$:

$$
\begin{aligned}
p &= m_0 \gamma \beta c \\
\frac{dp}{d\beta} &= m_0 c \frac{d}{d\beta}(\gamma\beta) \\
&= m_0 c\gamma + m_0 c\beta \frac{d\gamma}{d\beta} \\
&= m_0 c\gamma \left(1 + \beta^2\gamma^2\right) \\
&= m_0 c\gamma^3 \\
\Longrightarrow \frac{dp}{p} &= \gamma^2 \frac{d\beta}{\beta}
\end{aligned}
$$

We can now put together the two intermediate results and insert the definition of the momentum compaction factor:

$$\begin{aligned}\frac{\Delta T}{T} &= \frac{\Delta L}{L} - \frac{\Delta\beta}{\beta} \\ &= \alpha_c \cdot \frac{\Delta p}{p} - \frac{1}{\gamma^2} \cdot \frac{\Delta p}{p} \\ &= \left(\alpha_c - \frac{1}{\gamma^2}\right) \frac{\Delta p}{p}\end{aligned}$$

This result shows how the revolution period changes with momentum. There is a special energy, the so-called *transition energy* γ_{tr}, defined as:

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}}$$

At transition energy the revolution period becomes independent of the momentum spread and stays constant for off-momentum particles.

Assistant: Simon C. Leemann, simon.leemann@psi.ch, 056/310 3343