Introduction to Particle Accelerator Physics

Solutions to Exercise 2

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1. Thin Lens Approximation

$$\begin{array}{lclcl} \cos x & = & \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} & = & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots \\ \sin x & = & \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} & = & x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots \end{array}$$

$$M_Q = \begin{pmatrix} \cos\sqrt{k}L & \frac{1}{\sqrt{k}}\sin\sqrt{k}L \\ -\sqrt{k}\sin\sqrt{k}L & \cos\sqrt{k}L \end{pmatrix} = \begin{pmatrix} 1+\mathcal{O}(L^2) & \frac{\sqrt{k}L}{\sqrt{k}} + \mathcal{O}(L^3) \\ -kL + \mathcal{O}(L^3) & 1+\mathcal{O}(L^2) \end{pmatrix}$$

In the thin lens approximation we assume:

$$kL = \frac{1}{f} = \lim_{L\to 0}\int_0^{\infty} k(s)\,ds = const$$

$$\Longrightarrow \lim_{L\to 0} M_Q = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix}$$

2. Drift Sections and Quadrupole Doublets

a)

$$M_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$x = x_0 + Lx'_0 \quad \Longrightarrow \quad x'_0 = \frac{x - x_0}{L}$$

b)

$$M = M_{QD}M_DM_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{L}{f^2} & 1+\frac{L}{f} \end{pmatrix}$$

$$\Longrightarrow x = \left(1 - \frac{L}{f}\right)x_0 + Lx'_0$$

If we tune the quadrupole focusing strength so that $\frac{L}{f} = 1$ we get $x = Lx'_0$ which is independent of x_0.

c) From the lecture recall the definition of the quadrupole strength: $k = \frac{e}{p}g$ where g is the focusing gradient of the quadrupole. In 2b) we required $L = f$ and from the lecture we recall $\frac{1}{f} = k \cdot L$. Putting everything together we get:

$$g = \frac{p}{e}\cdot k = \frac{p}{e}\cdot\frac{1}{Lf} = \frac{p}{e}\cdot\frac{1}{L^2} = \frac{0.65\ \text{GeV/c}}{0.29979}\cdot\frac{1}{(0.931\ \text{mm})^2} = 2.5\ \text{T/m}$$

3. Stability Criterion for a Circular Accelerator
a)

$$M_{Rev} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

So here with $\alpha_0 = -\frac{\beta_0'}{2} = 0$, we get $\gamma_0 = -1/\beta_0$ and

$$M_{Rev} = \begin{pmatrix} \cos\mu & \beta_0 \sin\mu \\ -\frac{1}{\beta_0} \sin\mu & \cos\mu \end{pmatrix}$$

b) The betatron tune is defined as follows:

$$Q = \frac{\mu}{2\pi} = \frac{\phi_{Rev}}{2\pi}$$

where the total phase advance in one revolution (accelerator circumference C) is given by

$$\phi_{Rev} = \oint \frac{1}{\beta(s)}\, ds = \int_{s_0}^{s_0+C} \frac{1}{\beta(s)}\, ds$$

M_{Rev} looks like a rotation matrix for the rotation angle μ. The slight difference is the factor β_0 respectively $1/\beta_0$. This leads to an ellipse in phase space. A point on this ellipse advances by the angle μ. The tune Q is the total betatron phase advance μ divided by 2π. Therefore Q is the number of betatron oscillations per revolution.

c) We can easily calculate the trace $Tr(M_{Rev}) = 2\cos\mu$ and since we know that the cos function has the co-domain [-1,+1] in $\mathbb{R}$ we can derive a simple stability criterion:

$$-1 \leq \frac{Tr(M_{Rev})}{2} \leq +1$$

4. Transformation of the Beta Function
a) The beta matrix transforms like

$$B_s = M B_0 M^T$$

where

$$B_0 = \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix}$$

and

$$M = M_D = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Together this leads to

$$B_s = \begin{pmatrix} \beta_s & -\alpha_s \\ -\alpha_s & \gamma_s \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} & \frac{s}{\beta_0} \\ \frac{s}{\beta_0} & \frac{1}{\beta_0} \end{pmatrix}$$

So we end up with

$$\begin{aligned} \beta(s) &= \beta_0 + \tfrac{s^2}{\beta_0} \\ \alpha(s) &= -\tfrac{s}{\beta_0} \\ \gamma(s) &= \tfrac{1}{\beta_0} \end{aligned}$$

b)

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

where C denotes the cosine-like function and S denotes the sine-like function in the transformation matrix. In the thin lense approximation we can write

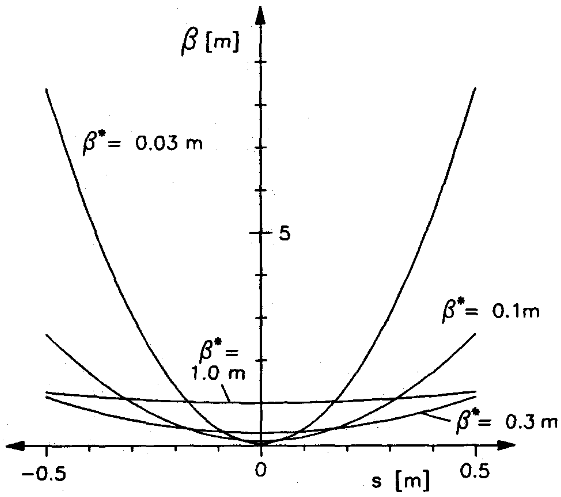
$$M = M_D = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

This then gives us

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -2s & s^2 \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} = \begin{pmatrix} \beta_0 + \frac{s^2}{\beta_0} \\ -\frac{s}{\beta_0} \\ \frac{1}{\beta_0} \end{pmatrix}$$

which is identical to the results we derived in 4a).

c) The betatron function $\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$ in the vicinity of a symmetry point in a drift section grows quadratically with the distance s from the symmetry point. This growth is larger for smaller values of β_0. This is a direct consequence of the Liouville's theorem: the area in phase space can not be reduced by a focussing element, thus a reduction of transverse beam size leads to a simultaneous increase of beam divergence. This is illustrated below ($\beta^\star$ denotes β_0).

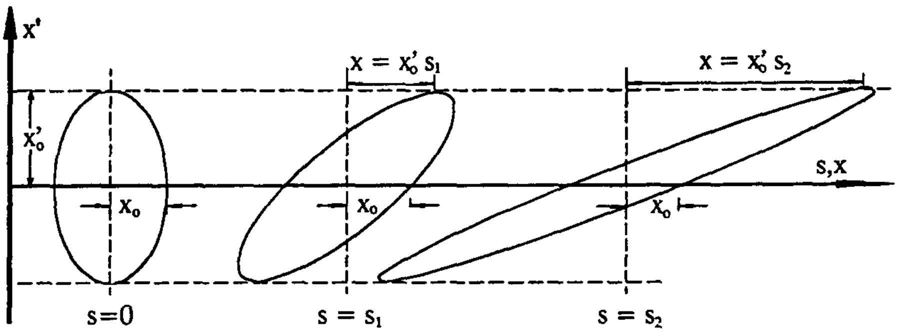


5. Evolution of a Phase Space Ellipse

a) From the transport matrix of a drift section we gather

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M_D \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 + x_0' \cdot s \\ x_0' \end{pmatrix}$$

So the divergence $x' = x_0'$ remains constant while the beam size $x = x_0 + x_0' \cdot s$ increases linearly with s.



b) The initially diverging beam becomes convergent after passing the focussing lens. It converges until it reaches the beam waist where the beam size reaches a minimum (and the divergence reaches a maximum). From here on the beam becomes divergent again.